Lecture 14

Thursday, 22 October 2026 10:37 PM

Last time:

Set N of n voters, A of m alternatives / candidates.
Each voter has a strict linear order $\succ_i \in L(A)$. (or preference / ranking)
Profile $P = (\succ_1, \dots, \succ_n)$
Social Choice Function $f : P \to A$
" " Correspondence $F : P \to 2^A$
" Welfare Function $F : P \to L(A)$

A SCF is truthful / strategyproof, if $\forall i, \succ_i, \succ_i', \succ_{-i}$,
$f(\succ_i, \succ_{-i}) \succsim_i f(\succ_i', \succ_{-i})$

G-S Theorem: For $m \geq 3$, every SCF that is onto, & truthful is a dictatorship.

f is strongly / Maskin **monotone** if: let $f(P) = a$, and P is an a-improvement of P. Then $f(P') = a$.

Claim: f is truthful $\Rightarrow$ f is monotone

Actually: For $m \geq 3$, every SCF that is onto & satisfies Maskin monotonicity is a dictatorship.

Single-Peaked Preferences:

Given a set of alternatives A, let $\rhd \in \mathcal{L}(A)$ be a linear order on A. — called the "base order"
A ranking $\succ$ is single-peaked wrt $\rhd$ if,
let p = top ranked alternative for $\succ$. Then:
(i) $p \trianglerighteq x \rhd y \Rightarrow x \succ y$
(ii) $y \rhd x \trianglerighteq p \Rightarrow x \succ y$

Thus p is the 'peak', & moving away from p makes a candidate worse.

Claim: If $\succ$ is single-peaked wrt $\rhd$, then the last-ranked alternative in $\succ$ is either the first- or the last-ranked alternative in $\rhd$.

Defn: A preference profile $P = (\succ_i)_{i \in N}$ is single-peaked wrt $\rhd$ if each ranking $\succ_i$ is single-peaked wrt $\rhd$.

Earlier: Given a profile P. $N(a,b) = |\{i : a \succ_i b\}|$
$\mu(a,b) = N(a,b) - N(b,a)$
a "beats" b if $\mu(a,b) > 0$
a is a Condorcet winner if $\forall b \neq a$, a beats b

Then: P:
$a \succ_1 b \succ_1 c$
$b \succ_2 c \succ_2 a$
$c \succ_3 a \succ_3 b$ has a cycle, hence no Condorcet winner.

But this is not a single-peaked profile, for any base order.

Recognizing single-peaked profiles:

Claim: A ranking $\succ_i$ is single-peaked wrt $\rhd$ iff $\forall k = 1 \dots m$, the top k alternatives in $\succ_i$ form an interval in $\rhd$.
(prove yourself)

C1Ps:
Let $A \in \{0,1\}^{m \times n}$. Then A has the "consecutive 1's property" (C1P) if $\exists$ a permutation of the columns, so that in each row, the 1s are consecutive.

Theorem (Booth & Lueker 1976): Given $A \in \{0,1\}^{m \times n}$, we can determine in polynomial time if A has the C1P, and if yes, obtain the required column permutation.
(w/o proof)

(note: A satisfies C1P $\Rightarrow$ A is TU)

Given a profile $P = (\succ_i)_{i \in N}$, consider the matrix $A \in \{0,1\}^{mn \times m}$ constructed as follows:
- column c_a $\forall a \in A$,
- row $r_{v,k}$ $\forall v \in V$, $k \in [m]$

The entry $A(r_{v,k}, c_a) = 1$ if $\text{rank}_v(a) \leq k$.

Theorem: Given a preference profile $P = (\succ_i)_{i \in N}$, P is single-peaked iff the matrix A satisfies the C1P, & the required permutation over the columns so that the 1s in each row are consecutive gives the required base order $\rhd$.

Proof: Suppose P is single-peaked for some base order $\rhd$. Order the columns (alternatives) according to $\rhd$. Consider any row $R_{v,k}$, this row has 1s for each of the top k alternatives of voter v. By the claim, these form an interval in $\rhd$, hence the 1s are contiguous.

Now suppose A satisfies C1P, & let $\rhd$ be the ordering so that in each row, the 1s are consecutive. Then for each voter v, & $k \in [m]$, the top k candidates form an interval in $\rhd$. By claim, $>_v$ is single-peaked wrt $\rhd$, & this holds for all v. ⊠

So why are single-peaked profiles useful?
(Black, 1948)

Theorem: Let P be single-peaked wrt $\rhd$, # voters n odd. Then:

① If a beats b, & b beats c, then a beats c

② Let $p_1, \dots, p_n$ be the top-ranked candidates for the voters, & let p_{med} be the median candidate according to $\rhd$. Then p_{med} is the Condorcet winner.

For each voter v, let p_v (the peak of v) be v's top-ranked alternative.

Proof: Will show first that p_{med} is the Condorcet winner.

Note that since n is odd, at least $(n+1)/2$ voters v have top-ranked alternatives (peaks) p_v s.t. $p_{med} \trianglerighteq p_v$, & similarly at least $(n+1)/2$ voters v have peaks p_v s.t. $p_v \trianglerighteq p_{med}$.

Now consider any $a \neq p_{med}$, say $a \rhd p_{med}$. But then for at least $\frac{n+1}{2}$ voters v, $a \rhd p_{med} \trianglerighteq p_v$, hence $p_{med} >_v a$,

& p_{med} beats a. Hence p_{med} beats every alternative, & is a Condorcet winner.

①. Fix any a, b, c, s.t. a beats b, b beats c. Restrict P to $\{a, b, c\}$, then this restricted profile is still single-peaked wrt $\rhd$ restricted to $\{a, b, c\}$ (check!) say $P|_{abc}$

By ②, there is a Condorcet winner in $P|_{abc}$. This has to be a, since a beats b, b beats c. But then a beats c also. ⊠

Thus, consider the directed graph $G = (A, E)$ where $\forall a, b \in A$, there is an edge (a, b) if a beats b, & (b, a) if b beats a (note that exactly one of these edges must exist). Call this graph the majority graph, & note that

(i) this is a tournament: there is a directed edge for every pair of vertices, and

(ii) by the theorem, the graph is acyclic, & is hence a DAG.

For alternatives A, fix a base order $\rhd$. Let $SP(\rhd)$ be the set of all single-peaked rankings wrt to $\rhd$. Suppose all true preferences of the voters are in $SP(\rhd)$.

Consider the SCF $f_{med} : SP(\rhd)^n \to A$, defined as follows:

Median Rule: Given a preference profile P, single-peaked wrt $\rhd$, Let $p_1, \dots, p_n$ be the peaks of the voters, and order the voters so that $p_1 \trianglerighteq p_2 \trianglerighteq \dots \trianglerighteq p_n$. Then: if n is odd, $p_{med} = p_{\frac{n+1}{2}}$, else $p_{med} = p_{\frac{n}{2}}$. f_{med} returns p_{med}.

Theorem: If real & reported preferences are in $SP(\rhd)$, then f_{med} is onto, non-dictatorial & strategyproof

Proof: Onto: $\forall a \in A$, if all voters report a as a peak, then f_{med} returns a.

Non-dictatorial: Suppose $a_1 \rhd \dots \rhd a_m$. Fix voter v. Let a_m be peak for voter v, & a_1 be the peak for all other voters. Then f_{med} returns a_1.

Strategyproof: Let p_v be v's true peak & P be the preference profile where v reports p_v truthfully, & p_{med} be the output. If $p_{med} = p_v$, then v already gets her best outcome. Now suppose $\boxed{p_{med} \rhd p_v.}$

For any report a s.t. $p_{med} \rhd a$, the median of the peaks does not change. For any report a s.t. $a \rhd p_{med}$, the new median $p'_{med} \trianglerighteq p_{med}$. But this is worse for v, hence misreporting cannot improve the outcome for v. ⊠

Generalized median:

Fix alternatives A, base ordering $\rhd$. Fix $n+1$ alternatives $\alpha = (\alpha_1, \dots, \alpha_{n+1})$ from A. (repetitions allowed).

The generalized median voting rule is

$$f_\alpha(P) = \text{median}(p_1, \dots, p_n, \alpha_1, \dots, \alpha_{n+1})$$

Let l & r be the first & last alternative in $\rhd$.

Thus: $\forall i$ $\alpha_i = l / r$

$f_\alpha(P)$ returns l / r

n odd:

$i \leq \frac{n+1}{2}$: $\alpha_i = l$, $i > \frac{n+1}{2}$ $\alpha_i = r$

$f_\alpha(P)$ returns median of P

n even:

$i \leq \frac{n}{2}+1$: $\alpha_i = l$, $i > \frac{n}{2}+1$, $\alpha_i = r$

$f_\alpha(P)$ returns the left median of P.

Theorem: Every generalized median rule is strategyproof on $SP(\rhd)^n$.